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CONSTITUTIVE EQUATIONS FOR A CLASS OF NONLINEAR ELASTIC SOLIDS*

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Abstract—Constitutive equations are developed for elastic solids sustaining deformation for which displacement gradients are small but where complete physical nonlinearity is permitted. The constitutive equation includes as special cases forms considered by recent authors; at the same time, more general effects are incorporated, in particular, coupling between volumetric and deviatoric components of strain.

Some simple states of deformation are examined and the plane elastostatic problem is formulated, together with the solution of an example problem by perturbation techniques.

INTRODUCTION

A NUMBER of materials of interest to engineers exhibit nonlinear mechanical effects, even when sustaining small deformations. Examples are materials—such as concrete, rock, solid propellants and foamed elastomers—whose tensile and compressive responses often differ and whose behavior is strongly dependent on superimposed hydrostatic stress. Another example is sand, which dilates when subjected to a state of simple shearing stress. Although this type of nonlinearity is a special case of the general nonlinear theory of elasticity [1], rather than simplify results which are complicated by simultaneous consideration of physical and kinematic nonlinearity, it has been more convenient to introduce kinematic restrictions initially and to develop the theory from this viewpoint. Accordingly, one is led to examine a theory of elastic solids for which kinematic linearity is retained but where physical nonlinearity is permitted.

The first significant contribution to physically nonlinear elasticity theory appears to have been in 1894 by Voigt [2], who extended the stress–strain law of Hooke–Cauchy to include quadratic terms in strain and thus developed a five constant elasticity theory, applying it to the solution of simple problems. The same form of law was considered by Murnaghan [3] and Biot [4]. Sternberg [5] applied the five constant theory to the extension of a rod and to torsion of a circular cylinder. Novozhilov [6] pointed out the restrictive nature of Voigt's five constant theory in its application to the behavior of most real materials and discussed a constitutive law retaining linear and cubic terms in strain, having six elastic constants.

The first application to boundary value problems appears in the work of Kauderer [7], who used perturbation techniques to obtain approximate solutions to a number of problems, although the form of constitutive law chosen, as will be seen, is quite restrictive. Recently, Savin [8] has used the restrictive form of constitutive theory due to Kauderer to obtain an approximate solution method for the extension of an infinite plate with a

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hole and has applied this method to specific problems [9]. A coupled thermoelastic theory where nonlinearity is present with respect to mechanical as well as thermal variables has been considered by Dillon [10] and, in a more general context, by Herrmann [11]. A special class of viscoelastic materials has been considered by Rivlin [12] and by Bergen *et al.* [13] with a constitutive law that is equivalent, for a given class of deformations, to a sub-class of that considered here.

It should be noted that the present definition of physical nonlinearity incorporates Reiner's definitions of physical and tensorial nonlinearity [14].

The present paper develops in the first section the most general form of constitutive law for isotropic solids, with the kinematic restrictions cited above. Features of the previous works mentioned then appear as special cases when suitable restrictions are placed on the form of the constitutive equation. In the second section, some simple states of deformation are investigated, through which the significance of the material property functions is made to appear. Likewise, the simplest type of coupling between volumetric and distortional effects is illustrated. Finally, the formulation of the plane elastostatic problem is treated, together with an approximate solution of the problem of uniform stretching of an infinite plate containing a circular hole.

1. FORMULATION OF THE CONSTITUTIVE LAW FOR HOMOGENEOUS ISOTROPIC SOLIDS

The mechanical constitutive equations for an elastic solid in the sense of Green can be written*

$$\tau_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} \tag{1.1}$$

where τ_{ij} and ε_{ij} are, respectively, the stress and strain tensors and U is the strain energy density. As stated above, the purpose of the present paper is to develop constitutive equations for which complete kinematic linearity is retained while permitting arbitrary physical nonlinearity. Accordingly, no distinction between material and spatial reference frames is required, and the linearized strain–displacement equations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{1.2}$$

apply, limiting deformation to the class of small strain, small rotation [6]. In (1.2) u_i are the components of the displacement vector. For a homogeneous, isotropic solid, the strain energy density can be expressed as

$$U = U(I_1, I_2, I_3) \tag{1.3}$$

where I_i, independent invariants of the strain tensor, are here taken to be

$$\begin{aligned} I_1 &= \varepsilon_{kk}, \\ I_2 &= \tfrac{1}{2}\varepsilon_{km}\varepsilon_{km}, \\ I_3 &= \tfrac{1}{3}\varepsilon_{km}\varepsilon_{kn}\varepsilon_{mn}. \end{aligned} \tag{1.4}$$

* Latin indices take on values 1, 2 or 3. Tensor notation is used and, for convenience, quantities are referred to rectangular Cartesian coordinates, Z_i, unless otherwise specified.

From (1.2) and (1.3)

$$\tau_{ij} = \frac{\partial U}{\partial I_1}\frac{\partial I_1}{\partial \varepsilon_{ij}} + \frac{\partial U}{\partial I_2}\frac{\partial I_2}{\partial \varepsilon_{ij}} + \frac{\partial U}{\partial I_3}\frac{\partial I_3}{\partial \varepsilon_{ij}}$$

or

$$\tau_{ij} = \phi_1 \delta_{ij} + \phi_2 \varepsilon_{ij} + \phi_3 \varepsilon_{ik}\varepsilon_{jk}, \tag{1.5}$$

where

$$\phi_i = \phi_i(I_j) = \frac{\partial U}{\partial I_i}. \tag{1.6}$$

From (1.6), the functions ϕ_i (material strain functions) are related by three equations

$$\frac{\partial \phi_i}{\partial I_j} = \frac{\partial \phi_j}{\partial I_i}. \tag{1.7}$$

The constitutive law (1.5) may also be derived from the theory of isotropic matrices [15] without assuming the existence of U. The material strain functions, no longer restricted by (1.7), define a Cauçhy elastic solid.

It should be noted that the choice of independent strain invariants appearing in (1.3) and (1.4) is arbitrary. The particular advantage of the choice here is the separation of three separate material functions ϕ_i in a simple manner [16]. (Compare with the form commonly used in finite elasticity [1].)

As an example of a typical constitutive equation, let U be written as a polynomial in I_i. If terms in U from second to fourth order in strain are retained, the "cubic" stress–strain law is obtained:

$$\begin{aligned}\tau_{ij} = {} & A_{11}\varepsilon_{kk}\delta_{ij} + A_{12}\varepsilon_{ij} + A_{21}\varepsilon_{kk}^2\delta_{ij} + A_{22}\varepsilon_{km}\varepsilon_{km}\delta_{ij} \\ & + 2A_{22}\varepsilon_{kk}\varepsilon_{ij} + A_{23}\varepsilon_{ik}\varepsilon_{jk} + A_{31}\varepsilon_{kk}^3\delta_{ij} + A_{32}\varepsilon_{km}\varepsilon_{km}\varepsilon_{ij} \\ & + A_{33}\varepsilon_{km}\varepsilon_{km}\varepsilon_{nn}\delta_{ij} + A_{33}\varepsilon_{kk}^2\varepsilon_{ij} + A_{34}\varepsilon_{kk}\varepsilon_{ip}\varepsilon_{jp} \\ & + 3A_{34}\varepsilon_{km}\varepsilon_{kn}\varepsilon_{mn}\delta_{ij}.\end{aligned} \tag{1.8}$$

Terms are grouped in order in (1.8), and it is readily seen that

$$\begin{aligned}\phi_1 &= A_{11}I_1 + A_{21}I_1^2 + 2A_{22}I_2 + A_{31}I_1^3 + 2A_{33}I_1I_2 + A_{34}I_3, \\ \phi_2 &= A_{12} + 2A_{22}I_1 + 2A_{32}I_2 + A_{33}I_1^2, \\ \phi_3 &= A_{23} + A_{34}I_1.\end{aligned} \tag{1.9}$$

It is to be noted that for a nonlinear solid at small strain, reduction to a linear law for $\varepsilon_{ij} \to 0$ need not be required.

There is, furthermore, no *a priori* reason for requiring that terms of a prescribed order be present in all material functions. The only reason for doing so here is to examine the most general polynomial law of a given order. In some instances it may be advantageous to expand the material functions appearing in (1.6) in power series of the invariants I_i; in such instances, the relations (1.7) must be observed. Finally, before passing to an examination of some special cases, it should be mentioned that anisotropy and nonhomogeneity can be treated with no further formal difficulty.

Incompressible elastic solids

For incompressible media,

$$I_1 = 0$$

and

$$U = U(I_2, I_3).$$

Introducing a Lagrangian multiplier, $-p$, (1.2) becomes

$$\tau_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} + (-p)\frac{\partial I_1}{\partial \varepsilon_{ij}} \tag{1.10}$$

and, from (1.10),

$$\tau_{ij} = -p\delta_{ij} + \phi_2 \varepsilon_{ij} + \phi_3 \varepsilon_{ik}\varepsilon_{jk}. \tag{1.11}$$

ϕ_2 and ϕ_3 are as defined by (1.6), and (1.7) gives the single equation

$$\frac{\partial \phi_2}{\partial I_3} = \frac{\partial \phi_3}{\partial I_2}.$$

p has dimensions of stress and in certain cases has physical significance as hydrostatic pressure.

The inverse constitutive law

Consider the scalar function, C, the complementary energy density, dependent only on the current state of stress such that

$$\varepsilon_{ij} = \frac{\partial C}{\partial \tau_{ij}}.$$

Then, by arguments identical to those used for the strain energy density function, for a homogeneous, isotropic solid

$$C = C(\theta_1, \theta_2, \theta_3),$$

where θ_i, the invariants of the stress tensor, are given by

$$\begin{aligned} \theta_1 &= \tau_{kk}, \\ \theta_2 &= \tfrac{1}{2}\tau_{km}\tau_{km}, \\ \theta_3 &= \tfrac{1}{3}\tau_{km}\tau_{kn}\tau_{mn}. \end{aligned} \tag{1.12}$$

The inverse constitutive law

$$\varepsilon_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \tau_{ij} + \alpha_3 \tau_{ik}\tau_{jk} \tag{1.13}$$

is thus obtained, where

$$\alpha_i = \alpha_i(\theta_j) = \frac{\partial C}{\partial \theta_i}. \tag{1.14}$$

From (1.14) it follows that the material stress functions, α_i, are related by three equations

$$\frac{\partial \alpha_i}{\partial \theta_j} = \frac{\partial \alpha_j}{\partial \theta_i}. \tag{1.15}$$

The existence of the complementary energy density function follows from the existence of the strain energy density function in that C is the Legendre transformation of U, the two functions being related by

$$C = \tau_{ij}\varepsilon_{ij} - U. \tag{1.16}$$

The condition that (1.5) has a unique inverse (1.13) is that the Hessian determinant of U is non vanishing, i.e.

$$H(U) \equiv \text{DET}\left|\frac{\partial^2 U}{\partial\varepsilon_{ij}\,\partial\varepsilon_{km}}\right| \neq 0. \tag{1.17}$$

The condition (1.17), which is identical to the non-vanishing of the Jacobian of $\tau_{ij}(\varepsilon_{km})$, is taken as a restriction on the material strain functions. It can be shown [17] that the restriction, together with the requirement that U be non-negative, is equivalent to Drucker's postulate of stability [18]. Equation (1.16) may be used as a starting point for obtaining inverse constitutive laws. Suppose that the form of (1.5) and hence of U is known. Since, from (1.5),

$$\tau_{ij}\varepsilon_{ij} = \phi_1 I_1 + 2\phi_2 I_2 + 3\phi_3 I_3, \tag{1.18}$$

(1.16) may be used to obtain $C(I_i)$. From (1.5) and making use of the Cayley–Hamilton theorem, $\theta_i(I_j)$ are obtained, the relation being

$$\begin{aligned}
\theta_1 &= 3\phi_1 + \phi_2 I_1 + 2\phi_3 I_2,\\
\theta_2 &= \tfrac{3}{2}\phi_1^2 + \phi_1\phi_2 I_1 + (\phi_2^2 + 2\phi_1\phi_3)I_2 + 3\phi_2\phi_3 I_3\\
&\quad + \phi_3^2(\tfrac{1}{12}I_1^4 - I_1^2 I_2 + I_2^2 + 2I_1 I_3),\\
\theta_3 &= \phi_1^3 + \phi_1^2\phi_2 I_1 + 2(\phi_1\phi_2^2 + \phi_1^2\phi_3)I_2 + (\phi_2^3 + 6\phi_1\phi_2\phi_3)I_3\\
&\quad + (\phi_1\phi_3^2 + \phi_2^2\phi_3)(\tfrac{1}{6}I_1^4 - 2I_1^2 I_2 + 2I_2^2 + 4I_1 I_3)\\
&\quad + \phi_2\phi_3^2(\tfrac{1}{6}I_1^5 + \tfrac{5}{3}I_1^3 I_2 + \tfrac{5}{2}I_1^2 I_3 + 5I_2 I_3)\\
&\quad + \tfrac{1}{3}\phi_3^3(\tfrac{1}{12}I_1^6 + \tfrac{1}{2}I_1^4 I_2 - 3I_1^2 I_2^2 + I_1^3 I_3 + 2I_2^3 + 3I_3^2 + 6I_1 I_2 I_3).
\end{aligned} \tag{1.19}$$

$C(\theta_i)$ is then computed.

As Truesdell (*ibid*), who uses a different but equivalent approach based on isotropic functions, points out, the analysis in general will be purely formal. However, if (1.5) is known in polynomial form, (1.13) may be obtained as a power series in θ_i by using (1.19) and equating coefficients of the power series $I_1^L I_2^M I_3^N$. Particular forms of (1.13) are analogous to those of (1.5), the "cubic" law, for instance, being as (1.8) with τ_{ij} replaced by ε_{ij} and the elastic moduli A_{AB} replaced by inverse moduli B_{AB}.

Further restrictions on the material functions

The material functions, ϕ_i, are restricted by (1.7) and (1.17). Further restrictions follow from considerations similar to those examined by Truesdell (*ibid*) and by Baker and Ericksen [19] for general nonlinear elastic solids:

(a) A zero state of stress must correspond to a zero state of strain (for compressible media).

(b) The greatest principal strain occurs in the direction of greatest principal stress.

Implications of these restrictions on both ϕ_i and α_i are the same as those discussed in the references cited.

Special classes of constitutive laws

There are sub-classes of the constitutive laws (1.5) and (1.13) which have been considered previously or which may describe the behavior of special classes of materials. Two particular sub-classes are now considered.

(a) *Materials for which the strain energy function does not depend on* I_3

If

$$U = U(I_1, I_2), \tag{1.20}$$

then

$$\phi_3 = 0 \tag{1.21}$$

and (1.5) reduces to

$$\tau_{ij} = \phi_1\delta_{ij} + \phi_2\varepsilon_{ij}. \tag{1.22}$$

Such a constitutive law was considered by Wainwright (*ibid*) in application to a physically nonlinear thin shell theory and by Dong [20] for viscoelastic solids. In applications to viscoelasticity, of course, ϕ_i depend on time as well as on the state of strain.

For incompressible solids, (1.22) has the form

$$\tau_{ij} = -p\delta_{ij} + \phi_2\varepsilon_{ij}.$$

This form of law was employed by Bergen *et al.* (*ibid*) for a restricted class of deformations of viscoelastic solids for which the material function could be expressed as a product of a function of time alone and a function of strain. Thus, the problem was pseudo-elastic in nature.

Reduction of the constitutive law to the form (1.22) is valid only where experimental evidence shows the condition to be true, e.g. [13]. The form of (1.5) cannot be simplified by geometric arguments.

When (1.21) holds, then the inverse law (1.13) may be similarly simplified, i.e.

$$\alpha_3 = 0 \tag{1.23}$$

and

$$\varepsilon_{ij} = \alpha_1\delta_{ij} + \alpha_2\tau_{ij}. \tag{1.24}$$

A proof of this follows:

When $\phi_3 = 0$, it is seen from (1.18) and (1.19) that

$$\tau_{ij}\varepsilon_{ij} = \tau_{ij}\varepsilon_{ij}(I_1, I_2) \tag{1.25}$$

and that

$$\begin{aligned} \theta_1 &= \theta_1(I_1, I_2), \\ \theta_2 &= \theta_2(I_1, I_2), \\ \theta_3 &= \theta_3(I_1, I_2, I_3). \end{aligned} \tag{1.26}$$

From (1.16), making use of (1.20) and (1.25),

$$C(I_i) = C(I_1, I_2). \tag{1.27}$$

If the complementary energy density is now expressed in terms of stress invariants, it follows from (1.26) and (1.27) that C cannot depend on θ_3 since θ_3, in turn, depends on I_3.

Hence,

$$C(\theta_i) = C(\theta_1, \theta_2)$$

and (1.23) and (1.24) follow.

(b) *Materials for which hydrostatic stress depends only on volumetric strain*

A class of materials which may be of interest is that for which the hydrostatic stress, τ_{kk}, is a function of the volumetric strain, ε_{kk}, i.e.

$$\theta_1 = \theta_1(I_1). \tag{1.28}$$

The form taken by (1.5) to satisfy this condition is obtained as follows:

Rewriting the first of (1.19),

$$\theta_1 = 3\phi_1 + I_1\phi_2 + 2I_2\phi_3. \tag{1.29}$$

With a view to satisfying (1.28), let

$$\phi_1 = f_1(I_1) + f_2(I_1, I_2, I_3). \tag{1.30}$$

From (1.28), (1.29) and (1.7), f_2, ϕ_2, and ϕ_3 must satisfy the four conditions

$$\begin{aligned} 3f_2 + I_1\phi_2 + 2I_2\phi_3 &= 0, \\ \frac{\partial f_2}{\partial I_2} &= \frac{\partial \phi_2}{\partial I_1}, \\ \frac{\partial f_2}{\partial I_3} &= \frac{\partial \phi_3}{\partial I_1}, \\ \frac{\partial \phi_2}{\partial I_3} &= \frac{\partial \phi_3}{\partial I_2}. \end{aligned} \tag{1.31}$$

Conditions (1.31) can be satisfied only if f_2, ϕ_2 and ϕ_3 are of the form

$$\begin{aligned} f_2 &= \tfrac{1}{3}I_1 f_3(I_2 - \tfrac{1}{6}I_1^2), \\ \phi_2 &= f_3(I_2 - \tfrac{1}{6}I_1^2), \\ \phi_3 &= 0. \end{aligned} \tag{1.32}$$

Thus, from (1.30) and (1.32), it follows that, if the constitutive law satisfies the condition (1.28), it has the form

$$\tau_{ij} = [f_1(\varepsilon_{kk}) - \tfrac{1}{3}\varepsilon_{kk} f_3(\varepsilon_{km}\varepsilon_{km} - \tfrac{1}{3}\varepsilon_{kk}^2)]\delta_{ij} + f_3(\varepsilon_{km}\varepsilon_{km} - \tfrac{1}{3}\varepsilon_{kk}^2)\varepsilon_{ij}. \tag{1.33}$$

Equation (1.33) is the form of constitutive law used by Kauderer (*ibid*) who obtained it by assuming, in addition to (1.28), that the deviatoric stress was related to deviatoric strain through the second invariant of deviatoric strain. From (1.33) it is readily seen that

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = [\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}] f_3(\varepsilon_{km}\varepsilon_{km} - \tfrac{1}{3}\varepsilon_{kk}^2). \tag{1.34}$$

If f_1 and f_3 are analytic, (1.33) may be expressed as

$$\tau_{ij} = \left[\sum_{n=1}^{N} a_n\varepsilon_{kk}^n - \tfrac{1}{3}\varepsilon_{kk}\sum_{n=0}^{M} b_n(\varepsilon_{km}\varepsilon_{km} - \tfrac{1}{3}\varepsilon_{kk}^2)^n\right]\delta_{ij} + \sum_{n=0}^{M} b_n(\varepsilon_{km}\varepsilon_{km} - \tfrac{1}{3}\varepsilon_{kk}^2)^n\varepsilon_{ij}. \tag{1.35}$$

If (1.5) is further restricted so that deviatoric stress is a function of deviatoric strain only, i.e.

$$\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij} = f(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}),$$

then from (1.34), f_3 must be a constant and the coefficients in (1.35) are restricted to

$$b_0 = b,$$
$$b_n = 0 \quad \text{if} \quad n \geqslant 1.$$

Then (1.5) becomes

$$\tau_{ij} = \sum_{n=1}^{M} a'_n(\varepsilon_{kk})^n\delta_{ij} + b\varepsilon_{ij} \tag{1.36}$$

where

$$a'_1 = a_1 - \tfrac{1}{3}b,$$
$$a'_n = a_n \quad \text{if} \quad n \geqslant 2.$$

Equation (1.36) describes behavior linear in shear and nonlinear in bulk response. It describes, for instance, the behavior of certain fibrous composites which are linear in shear but whose response in simple compression differs from that in simple tension even for very small strains.

2. SIMPLE STATES OF STRESS AND DEFORMATION

Examination of simple states of stress and deformation illustrates the physical phenomena appearing in the physically nonlinear constitutive theory discussed in the previous section. Left in general form involving material functions of strain or stress, the resulting expressions may be misleading since material functions are, in certain cases, isolated. The material functions, however, are themselves functions of the state of strain (or stress) and cannot be determined from a single test. Utilizing approximate representations of these functions (such as polynomials), it is possible to identify features arising from physical nonlinearity. A complete definition of the material functions (as functions of three independent variables) requires an elaborate experimental program.

Homogeneous states of stress

(a) *Simple tension:* $\tau_{11} = T$; $\tau_{ij} = 0$, $i \neq 1$, $j \neq 1$

It is convenient to use (1.13) as the constitutive law. Then,

$$\begin{aligned} \theta_1 &= T, \\ \theta_2 &= \tfrac{1}{2}T^2, \\ \theta_3 &= \tfrac{1}{3}T^3, \end{aligned} \tag{2.1}$$

and strains are given by

$$\begin{aligned}\varepsilon_{11} &= \alpha_1+\alpha_2 T+\alpha_3 T^2,\\ \varepsilon_{22} &= \varepsilon_{33} = \alpha_1,\\ \varepsilon_{12} &= \varepsilon_{13} = \varepsilon_{23} = 0.\end{aligned} \tag{2.2}$$

For a complete quadratic law, for instance,

$$\begin{aligned}\alpha_1 &= B_{11}T+B_{21}T^2+B_{22}T^2,\\ \alpha_2 &= B_{12}+2B_{22}T,\\ \alpha_3 &= B_{23}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\varepsilon_{11} &= (B_{11}+B_{12})T+(B_{21}+3B_{22}+B_{23})T^2,\\ \varepsilon_{22} &= \varepsilon_{33} = B_{11}T+(B_{21}+B_{22})T^2.\end{aligned} \tag{2.4}$$

It is to be noted that by using (1.13) instead of (1.5) and appealing to the invertibility condition (1.17) the ambiguity connected with existence and uniqueness of solution in the simple tension test (as arises in general nonlinear theory) is avoided. Note also that the extensional strain is an unsymmetric function of stress.

(b) *Simple shear:* Let $\varepsilon_{12} = \varepsilon$ be the only non-vanishing component of strain.

Using the constitutive law (1.5),

$$\begin{aligned}I_1 &= 0,\\ I_2 &= \varepsilon^2,\\ I_3 &= 0\end{aligned}$$

and, retaining up to cubic terms in the constitutive equation (1.8),

$$\begin{aligned}\tau_{12} &= \phi_2\varepsilon = (A_{12}+2A_{32}\varepsilon)\varepsilon,\\ \tau_{11} &= \tau_{22} = \phi_1+\phi_3\varepsilon^2 = (2A_{22}+A_{23})\varepsilon^2,\\ \tau_{33} &= \phi_1 = 2A_{22}\varepsilon^2,\\ \tau_{13} &= \tau_{23} = 0.\end{aligned} \tag{2.5}$$

In general, for this state of strain, ϕ_1 and ϕ_3 will not be zero and consequently, in order to maintain simple shear deformation, normal stresses must be applied. From (2.5) it is clear that the constant A_{12} is identified with the linear shear modulus while the remaining constants are associated with nonlinear response. The dual of simple shearing deformation (i.e. simple shearing stress) leads to the result that simple shearing stress is accompanied by dilatation. Starting from (1.13) it is easily shown that

$$\varepsilon_{kk} = \frac{\Delta V}{V} = 3\alpha_1+2\alpha_3\tau^2. \tag{2.6}$$

For a cubic approximation of the constitutive equation

$$\frac{\Delta V}{V} = 2(3B_{22}+B_{23})\tau^2. \tag{2.7}$$

These states of stress illustrate the difficulty mentioned above regarding experimental determination of the material functions. For example, although the relationship between shear stress and shear strain involves only the material function ϕ_2, this does not mean that it would be possible to determine ϕ_2 from a shear test alone. Since, for simple shear,

$$I_1 = I_2 = 0,$$

$\phi_2(I_1, I_2, I_3)$ could only be determined to the extent of its dependence on I_2.

(c) *Combined hydrostatic stress and shear stress*

Let

$$\tau_{12} = \tau, \qquad \tau_{11} = \tau_{22} = \tau_{33} = \tfrac{1}{3}p, \qquad \tau_{13} = \tau_{23} = 0.$$

Using (1.13)

$$\begin{aligned} \theta_1 &= p, \\ \theta_2 &= \tfrac{1}{6}p^2 + \tau^2, \\ \theta_3 &= p(\tfrac{1}{27}p^2 + \tfrac{2}{3}\tau^2), \end{aligned} \tag{2.8}$$

and strains are given by

$$\begin{aligned} \varepsilon_{11} = \varepsilon_{22} &= \alpha_1 + \tfrac{1}{3}\alpha_2 p + \alpha_3(\tfrac{1}{9}p^2 + \tau^2), \\ \varepsilon_{33} &= \alpha_1 + \tfrac{1}{3}\alpha_2 p + \tfrac{1}{9}\alpha_3 p^2, \\ \varepsilon_{12} &= \alpha_2 \tau + \tfrac{1}{3}\alpha_3 p\tau, \\ \varepsilon_{13} &= \varepsilon_{23} = 0. \end{aligned} \tag{2.9}$$

Using the complete quadratic law,

$$\begin{aligned} \alpha_1 &= B_{11}p + B_{21}p^2 + 2B_{22}(\tfrac{1}{6}p^2 + \tau^2), \\ \alpha_2 &= B_{12} + 2B_{22}p, \\ \alpha_3 &= B_{23}, \end{aligned} \tag{2.10}$$

whence

$$\begin{aligned} \varepsilon_{11} = \varepsilon_{22} &= (B_{11} + \tfrac{1}{3}B_{12})p + (B_{21} + \tfrac{5}{6}B_{22} + \tfrac{1}{9}B_{23})p^2 + (B_{22} + B_{23})\tau^2, \\ \varepsilon_{33} &= (B_{11} + \tfrac{1}{3}B_{12})p + (B_{21} + \tfrac{5}{6}B_{22} + \tfrac{1}{9}B_{23})p^2 + B_{22}\tau^2, \\ \varepsilon_{12} &= B_{12}\tau + (2B_{22} + \tfrac{1}{3}B_{23})p\tau. \end{aligned} \tag{2.11}$$

This state of stress illustrates the coupling which, in general, exists between bulk and deviatoric effects and is particularly significant for filled heterogeneous materials. For example, in $(2.11)_3$ the shear strain–stress law is seen to be an unsymmetric function of the hydrostatic stress. Evaluation of the constants appearing in (2.11) could be carried out by conducting triaxial tests of the type commonly employed in solid propellant and soil mechanics experiments.

3. PLANE ELASTOSTATIC BOUNDARY VALUE PROBLEMS

In this section the implications of physical nonlinearity on formulation and solution of two-dimensional boundary value problems are discussed. In what follows it will be assumed that, for the generalized plane stress problem, dependent variables have been averaged by integration over the small thickness of the solid plate.

By redefining the material functions ϕ_i and α_i, the constitutive law for plane problems may be simplified from the forms (1.5) or (1.13) to forms similar to those of (1.22) and (1.24).

For the case of plane strain, the only non-vanishing components of strain are ε_{11}, ε_{12} and ε_{22}.

From (1.4), the strain invariants become

$$\begin{aligned} I_1 &= \varepsilon_{11} + \varepsilon_{22}, \\ I_2 &= \tfrac{1}{2}(\varepsilon_{11}^2+\varepsilon_{22}^2+2\varepsilon_{12}^2), \\ I_3 &= \tfrac{1}{3}(\varepsilon_{11}^3+\varepsilon_{22}^3+3\varepsilon_{12}^2(\varepsilon_{11}+\varepsilon_{22})) \end{aligned} \tag{3.1}$$

and from (3.1)

$$I_3 = I_1(I_2-\tfrac{1}{6}I_1^2). \tag{3.2}$$

Thus, if (3.2) is used in (1.3), the strain energy density function may be written as

$$U = U(I_1, I_2)$$

and

$$\tau_{ij} = \phi'_1\delta_{ij}+\phi'_2\varepsilon_{ij} \tag{3.3}$$

where the prime indicates that ϕ'_i is not the same function as ϕ_i.

As an example, if the "cubic" constitutive law as given in (1.8) were used in a plane strain problem, the material functions ϕ_i given by (1.9) could be replaced by ϕ'_1 and ϕ'_2 in (3.3) where

$$\begin{aligned} \phi'_1 &= A_{11}I_1+(A_{21}-\tfrac{1}{2}A_{23})I_1^2+(2A_{22}+A_{23})I_2 \\ &\quad +2(A_{33}+A_{34})I_1I_2+(A_{31}-\tfrac{2}{3}A_{34})I_1^3, \\ \phi'_2 &= A_{12}+(2A_{22}+A_{23})I_1I_2+2A_{32}I_2+(A_{33}+A_{34})I_1^2. \end{aligned}$$

As shown in the first section, a constitutive law of the form (3.3) has an inverse

$$\varepsilon_{ij} = \alpha'_1\delta_{ij}+\alpha'_2\tau_{ij}. \tag{3.4}$$

For generalized plane stress

$$\theta_3 = \theta_1(\theta_2-\tfrac{1}{6}\theta_1^2)$$

and by the same arguments as used above, the constitutive law may be simplified to the form (3.3) or (3.4). In the sequel, the primes on the material functions in (3.3) and (3.4) will be omitted with the understanding that the constitutive laws (1.5) and (1.13) have been reduced to the simplified form (3.3) and (3.4).

Direct formulation of the generalized plane stress problem

In plane problems, the only non-vanishing strain compatibility equation is

$$\varepsilon_{11,22}+\varepsilon_{22,11}-2\varepsilon_{12,12} = 0. \tag{3.5}$$

Substituting (3.4) into (3.5)

$$\alpha_{1,11}+\alpha_{1,22}+(\alpha_2\tau_{11})_{,22}+(\alpha_2\tau_{22})_{,11}-2(\alpha_2\tau_{12})_{,12} = 0. \tag{3.6}$$

Introducing the Airy stress function, Φ, through the equation*

$$\tau_{\alpha\beta} = \delta_{\alpha\beta}\nabla^2\Phi-\Phi_{,\alpha\beta} \tag{3.7}$$

identically satisfies the stress equilibrium equations. Substituting then from (3.7) into (3.6), the compatibility equation becomes

$$\alpha_{1,\beta\beta}+(\alpha_2\Phi_{,22})_{,22}+(\alpha_2\Phi_{,11})_{,11}+2(\alpha_2\Phi_{,12})_{,12} = 0$$

or

$$\alpha_{1,\beta\beta}+(\alpha_2\Phi_{,\alpha\beta})_{,\alpha\beta}. \tag{3.8}$$

Thus, for given material stress functions, α_1 and α_2, (3.8) may be expressed explicitly as a nonlinear fourth order homogeneous partial differential equation.

The invariants of stress in terms of Φ are

$$\begin{aligned}\theta_1 &= \Phi_{,\alpha\alpha},\\ \theta_2 &= \tfrac{1}{2}\Phi_{,\alpha\beta}\Phi_{,\alpha\beta}.\end{aligned} \tag{3.9}$$

As an example of (3.8), for the "cubic" strain–stress law,

$$\begin{aligned}\varepsilon_{ij} = {}& B_{11}\tau_{kk}\delta_{ij}+B_{12}\tau_{ij}+B_{21}\tau_{kk}^2\delta_{ij}+B_{22}\tau_{km}\tau_{km}\delta_{ij}\\ &+2B_{22}\tau_{kk}\tau_{ij}+B_{31}\tau_{kk}^3\delta_{ij}+B_{32}\tau_{km}\tau_{km}\delta_{ij}\\ &+B_{33}\tau_{km}\tau_{km}\tau_{nn}\delta_{ij}+B_{33}\tau_{kk}^2\tau_{ij},\end{aligned} \tag{3.10}$$

for which

$$\begin{aligned}\alpha_1 &= B_{11}\theta_1+B_{21}\theta_1^2+2B_{22}\theta_2+B_{31}\theta_1^3+2B_{33}\theta_1\theta_2,\\ \alpha_2 &= B_{12}+2B_{22}\theta_1+2B_{32}\theta_2+B_{33}\theta_1^2,\end{aligned} \tag{3.11}$$

(3.8) then takes the form

$$\begin{aligned}&(B_{11}+B_{12})\nabla^4\Phi+B_{21}\nabla^2[(\nabla^2\Phi)^2]\\ &\quad+B_{22}[\nabla^2(\Phi_{,\alpha\beta}\Phi_{,\alpha\beta})+2(\nabla^2\Phi\Phi_{,\alpha\beta})_{,\alpha\beta}]\\ &\quad+B_{31}\nabla^2[(\nabla^2\Phi)^3]+B_{32}(\Phi_{,\alpha\beta}\Phi_{,\alpha\beta}\Phi_{,\gamma\delta})_{,\gamma\delta}\\ &\quad+B_{33}[\nabla^2(\nabla^2\Phi\Phi_{,\alpha\beta}\Phi_{,\alpha\beta})+((\nabla^2\Phi)^2\Phi_{,\alpha\beta})_{,\alpha\beta}] = 0.\end{aligned} \tag{3.12}$$

Perturbation solution scheme for the compatibility equation

It is not possible, in general, to obtain a closed form solution for (3.8). If, however, the constitutive law is "close" to the linear law and is in a polynomial form, an approximate solution scheme may be generated by perturbing the linear solution. This solution scheme has been used by Kauderer (*ibid*) and by Savin (*ibid*) for the particular class of nonlinearity referred to earlier. Φ is expanded in terms of a characteristic parameter, α,

* Greek indices take on values 1, 2 only.

such that

$$\Phi = \alpha\Phi^{(1)} + \alpha^2\Phi^{(2)} + \alpha^3\Phi^{(3)} + \cdots. \tag{3.13}$$

Substituting from (3.13) into (3.8) and requring the coefficient of each power of α to vanish, a succession of linear differential equations is obtained. The coefficient of α^1 gives the compatibility equation associated with the linear problem while, for each successive power of α, there is obtained a differential equation which is the biharmonic equation with a forcing function dependent on the preceding solutions, i.e.

$$\begin{aligned}
&\nabla^4\Phi^{(1)} = 0,\\
&(B_{11}+B_{12})\nabla^4\Phi^{(2)} + F_1(\Phi^{(1)}) = 0,\\
&\quad\vdots\\
&(B_{11}+B_{12})\nabla^4\Phi^{(n)} + F_n(\Phi^{(1)}, \ldots \Phi^{(n-1)}) = 0.
\end{aligned} \tag{3.14}$$

It is convenient to take the solution for $\alpha\Phi^{(1)}$ to satisfy the actual boundary conditions whilst the functions $\Phi^{(n)}(n > 1)$ satisfy homogeneous boundary conditions.

To illustrate (3.14), for a material with constitutive law (3.10), the first three terms of (3.14) are

$$\begin{aligned}
&\nabla^4\Phi^{(1)} = 0,\\
&(B_{11}+B_{12})\nabla^4\Phi^{(2)} + B_{21}\nabla^2[(\nabla^2\Phi^{(1)})^2]\\
&\qquad + B_{22}[\nabla^2(\Phi^{(1)}_{,\alpha\beta}\Phi^{(1)}_{,\alpha\beta}) + 2(\nabla^2\Phi^{(1)}\Phi^{(1)}_{,\alpha\beta})_{,\alpha\beta}] = 0,\\
&(B_{11}+B_{12})\nabla^4\Phi^{(3)} + 2B_{21}\nabla^2(\nabla^2\Phi^{(1)}\nabla^2\Phi^{(2)})\\
&\qquad + B_{22}[2\nabla^2(\Phi^{(1)}_{,\alpha\beta}\Phi^{(2)}_{,\alpha\beta}) + (\nabla^2\Phi^{(1)}\Phi^{(2)}_{,\alpha\beta} + \Phi^{(1)}_{,\alpha\beta}\nabla^2\Phi^{(2)})_{,\alpha\beta}]\\
&\qquad + B_{31}\nabla^2[(\nabla^2\Phi^{(1)})^3] + B_{32}(\Phi^{(1)}_{,\alpha\beta}\Phi^{(1)}_{,\alpha\beta}\Phi^{(1)}_{,\gamma\delta})_{,\gamma\delta}\\
&\qquad + B_{33}[\nabla^2(\nabla^2\Phi^{(1)}\Phi^{(1)}_{,\alpha\beta}\Phi^{(1)}_{,\alpha\beta}) + ((\nabla^2\Phi^{(1)})^2\Phi^{(1)}_{,\alpha\beta})_{,\alpha\beta}] = 0.
\end{aligned} \tag{3.15}$$

An alternative approximate formulation of the plane problem

For generalized plane stress, the formulation (3.8) is convenient for the constitutive law (3.4); however, it is not useful for plane strain since τ_{33}, which occurs in (3.4), is non-zero. Furthermore, in the nonlinear case, the condition

$$\varepsilon_{33} = 0 \tag{3.16}$$

will not enable τ_{33} to be expressed, in closed form, in terms of $\tau_{\alpha\beta}$. An alternative approximate formulation for the problem [1] can be developed by expanding both strain and stress in powers of a characteristic parameter, α, i.e.

$$\begin{aligned}
\varepsilon_{ij} &= \alpha\varepsilon^{(1)}_{ij} + \alpha^2\varepsilon^{(2)}_{ij} + \alpha^3\varepsilon^{(3)}_{ij} + \cdots,\\
\tau_{ij} &= \alpha\tau^{(1)}_{ij} + \alpha^2\tau^{(2)}_{ij} + \alpha^3\tau^{(3)}_{ij} + \cdots.
\end{aligned} \tag{3.17}$$

For polynomial law (3.4), substituting from (3.17) and requiring the coefficient of each power of α to be zero, a series of constitutive laws is obtained, each having the form

$$
\begin{aligned}
\varepsilon_{ij}^{(1)} &= B_{11}\tau_{kk}^{(1)}\delta_{ij}+B_{12}\tau_{ij}^{(1)},\\
\varepsilon_{ij}^{(2)} &= B_{11}\tau_{kk}^{(2)}\delta_{ij}+B_{12}\tau_{ij}^{(2)}+F(\tau_{ij}^{(1)}),\\
&\vdots\\
\varepsilon_{ij}^{(p)} &= B_{11}\tau_{kk}^{(p)}\delta_{ij}+B_{12}\tau_{ij}^{(p)}+F'(\tau_{ij}^{(p-1)}\cdots\tau_{ij}^{(1)}).
\end{aligned}
\tag{3.18}
$$

The coefficients of the first three powers of α in (3.10), for example, give

$$
\begin{aligned}
\varepsilon_{ij}^{(1)} &= B_{11}\tau_{kk}^{(1)}\delta_{ij}+B_{12}\tau_{ij}^{(1)},\\
\varepsilon_{ij}^{(2)} &= B_{11}\tau_{kk}^{(2)}\delta_{ij}+B_{12}\tau_{ij}^{(2)}+B_{21}\tau_{kk}^{(1)2}\delta_{ij}\\
&\quad +B_{22}\tau_{km}^{(1)}\tau_{km}^{(1)}\delta_{ij}+2B_{22}\tau_{kk}^{(1)}\tau_{ij}^{(1)},\\
\varepsilon_{ij}^{(3)} &= B_{11}\tau_{kk}^{(3)}\delta_{ij}+B_{12}\tau_{ij}^{(3)}+2B_{21}\tau_{kk}^{(1)}\tau_{mm}^{(2)}\delta_{ij}\\
&\quad +2B_{22}(\tau_{km}^{(1)}\tau_{km}^{(2)}\delta_{ij}+\tau_{kk}^{(1)}\tau_{ij}^{(2)}+\tau_{ij}^{(1)}\tau_{kk}^{(2)})\\
&\quad +B_{31}\tau_{kk}^{(1)3}\delta_{ij}+B_{32}\tau_{km}^{(1)}\tau_{km}^{(1)}\tau_{ij}^{(1)}\\
&\quad +B_{33}(\tau_{km}^{(1)}\tau_{km}^{(1)}\tau_{nn}^{(1)}\delta_{ij}+\tau_{kk}^{(1)2}\tau_{ij}^{(1)})
\end{aligned}
\tag{3.19}
$$

To formulate the plane stress problem, the Airy stress function (3.7) together with (3.13) is used in the constitutive laws (3.18), which are then substituted into the compatibility equation (3.5). The resulting differential equations are, of course, precisely those given by (3.14). However, in the present formulation of the problem for plane strain, it is necessary to eliminate τ_{33} from each of the constitutive equations (3.18). To do this, the condition (3.16) is used which, from the first of (3.17), requires that

$$\varepsilon_{33}^{(n)} = 0, \qquad n = 1, 2, \ldots. \tag{3.20}$$

Using (3.20) in (3.18), the equivalent plane strain constitutive laws may be obtained. The equations (3.19), for example, are

$$
\begin{aligned}
\varepsilon_{\alpha\beta}^{(1)} &= \bar{B}_{11}\tau_{\gamma\gamma}^{(1)}\delta_{\alpha\beta}+\bar{B}_{12}\tau_{\alpha\beta}^{(1)},\\
\varepsilon_{\alpha\beta}^{(2)} &= \bar{B}_{11}\tau_{\gamma\gamma}^{(2)}\delta_{\alpha\beta}+\bar{B}_{12}\tau_{\alpha\beta}^{(2)}+\bar{B}_{21}\tau_{\gamma\gamma}^{(2)2}\delta_{\alpha\beta}\\
&\quad +\bar{B}_{22}\tau_{\gamma\delta}^{(1)}\tau_{\gamma\delta}^{(1)}\delta_{\alpha\beta}+2\bar{B}_{22}\tau_{\gamma\gamma}^{(1)}\tau_{\alpha\beta}^{(1)},\\
\varepsilon_{\alpha\beta}^{(3)} &= \bar{B}_{11}\tau_{\gamma\gamma}^{(3)}\delta_{\alpha\beta}+\bar{B}_{12}\tau_{\alpha\beta}^{(3)}+2\bar{B}_{21}\tau_{\gamma\gamma}^{(1)}\tau_{\delta\delta}^{(2)}\delta_{\alpha\beta}\\
&\quad +2\bar{B}_{22}(\tau_{\gamma\delta}^{(1)}\tau_{\gamma\delta}^{(2)}\delta_{\alpha\beta}+\tau_{\delta\delta}^{(1)}\tau_{\alpha\beta}^{(2)}+\tau_{\alpha\beta}^{(1)}\tau_{\delta\delta}^{(2)})\\
&\quad +\bar{B}_{31}\tau_{\delta\delta}^{(1)3}\delta_{\alpha\beta}+\bar{B}_{32}\tau_{\gamma\delta}^{(1)}\tau_{\gamma\delta}^{(1)}\tau_{\alpha\beta}^{(1)}\\
&\quad +\bar{B}_{33}(\tau_{\gamma\delta}^{(1)}\tau_{\gamma\delta}^{(1)}\tau_{\rho\rho}^{(1)}\delta_{\alpha\beta}+\tau_{\gamma\gamma}^{(1)2}\tau_{\alpha\beta}^{(1)}),
\end{aligned}
\tag{3.21}
$$

where

$$
\begin{aligned}
\bar{B}_{11} &= \frac{B_{12}}{(B_{11}+B_{12})}B_{11},\\
\bar{B}_{12} &= B_{12},\\
\bar{B}_{21} &= \frac{B_{12}}{(B_{11}+B_{12})^3}[B_{12}^2B_{21}-3B_{11}^2B_{22}],
\end{aligned}
$$

$$\bar{B}_{22} = \frac{B_{12}}{(B_{11}+B_{12})} B_{22},$$

$$\bar{B}_{31} = \frac{1}{(B_{11}+B_{12})^5} \{(B_{11}+B_{12})[B_{12}^4 B_{31} - B_{11}^4 B_{32} + 2B_{11}^2 B_{12}^2 B_{33}] - 2[B_{12}^2 B_{21} + B_{11}(B_{11}-2B_{12})B_{22}]^2\},$$

$$\bar{B}_{32} = B_{32} - \frac{2}{(B_{11}+B_{12})} B_{22}^2, \tag{3.22}$$

$$\bar{B}_{33} = \frac{1}{(B_{11}+B_{12})^3} \{(B_{11}+B_{12})[B_{11}^2 B_{32} + B_{12}^2 B_{33}] - 2[B_{12}^2 B_{21} + B_{11}(B_{11}-2B_{12})B_{22}]B_{22}\}.$$

Thus, by replacing the constants B_{AB} in the constitutive law by equivalent constants $\bar{B}_{AB}$, the plane strain problem may be considered as if it were plane stress and, as in the linear case, a given solution may be adapted to plane strain or plane stress provided the elastic constants are properly interpreted.

It is to be noted that, for nonlinear solids, the solution of plane problems, in general, will *not* be independent of the elastic constants.

From (3.22) it is seen that the cubic constants in plane strain will not be zero even if the cubic constants in the constitutive law are zero. The second perturbation of quadratic terms is thus effected both through the coefficients $\bar{B}_{2A}$ and $\bar{B}_{3A}$.

Example—the extension of an infinite plate containing a circular hole

Kauderer (*ibid*) obtained a number of approximate solutions to plane stress problems for a special class of nonlinearity. A single perturbation of quadratic terms was considered. Using complex variables, Savin (*ibid*) formulated the problem of the tension of an infinite plate containing a hole, again using a single perturbation of Kauderer's quadratic law. In the solution described here, the first and second perturbations are carried out for the "cubic" strain–stress law given by equation (3.10).

The equations solved are (3.15). These take into account the most general nonlinearity up to third order in the strain–stress law and, as shown above, may be used for plane stress or plane strain depending on the interpretation of the elastic constants.

Figure 1 shows part of the plate. The radius of the hole is a and a uniform tension, S, is applied at infinity. It is convenient to use polar coordinates r and θ located as shown. Since the equations (3.15) are in invariant form, they may be applied in curvilinear coordinates if the partial differentiation is replaced by appropriate invariant differentiation [21].

The solution method is straightforward and will not be discussed in any detail. The classical solution is well known [22] and provides the stress function $\Phi^{(1)}$. The result is then utilized in computing the forcing function in the governing equation for $\Phi^{(2)}$, this function subsequently being determined to satisfy homogeneous stress boundary conditions. $\Phi^{(3)}$ is determined in a similar manner. Of particular interest is the effect of nonlinearity on the stress concentration factor i.e.,

$$\tau_{\theta\theta}(r = a, \theta = \pi/2).$$

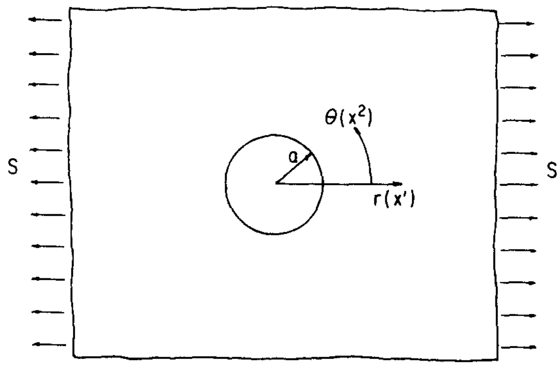


FIG. 1.

In terms of Φ,

$$\tau_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

and, computing this quantity, one obtains

$$\tau_{\theta\theta}\Big|_{\substack{r=a\\ \theta=\pi/2}} = S[3+\frac{S}{B}(-2B_{21}-9B_{22})+\frac{S^2}{B^2}(9{\cdot}3B_{21}^2+66B_{21}B_{22}+133B_{22}^2)$$
$$+\frac{S^2}{B}(-10B_{31}-14{\cdot}3B_{32}-25{\cdot}5B_{33})]$$

where

$$B = B_{11}+B_{12}$$

The general effect is as expected. As shown in Fig. 2, positive elastic constants, $B_{AB}(A > 1)$ indicate that the material softens under increasing uniaxial tensile load, and a corresponding reduction in stress concentration would be expected. The opposite effect, i.e. an increase in stress concentration, would occur for the same material subjected to uniaxial compression.

A feature brought out by the stress concentration is that the second perturbation for quadratic terms is not small compared with the first perturbation unless the deviation from linearity is very small.

As seen from Fig. 2(a), the ratios $B_{21}S/B$ and $3B_{22}S/B$ are measures of nonlinearity with respect to the uniaxial test and each ratio has a coefficient in the second perturbation of between four and five times its coefficient in the first perturbation. Thus, the magnitude of the stress concentration due to the second perturbation would be approximately the same as that due to the first perturbation for a nonlinearity of 20 per cent.

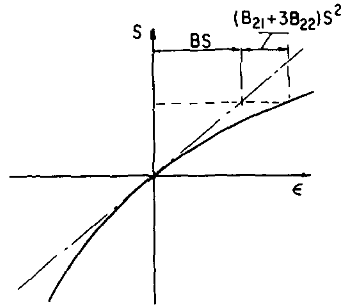
In order to consider more fully the convergence of the perturbation series, two further perturbations were carried out for the coefficient B_{21}.

Omitting computational details, for a constitutive law

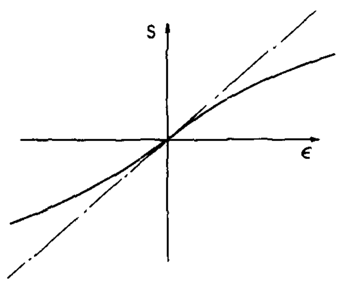
$$\varepsilon_{ij} = B_{11}\tau_{kk}\delta_{ij}+B_{12}\tau_{ij}+B_{21}\tau_{kk}^2\delta_{ij},$$

the stress concentration factor is

$$\tau_{\theta\theta}\Big|_{\substack{r=a\\ \theta=\pi/2}} = S[3-2k+9{\cdot}33k^2-50{\cdot}4k^3+297^4\ldots] \tag{3.23}$$



(a) "Quadratic" material, $B_{2A} > 0$



(b) "Cubic" material, $B_{3A} > 0$

FIG. 2.

where

$$k = B_{21}S/B$$

is the nonlinearity ratio with respect to uniaxial tension.

If an approximation of 10 per cent to the correction of stress concentration due to nonlinearity is taken as an acceptable criterion to terminate the perturbation series, one might infer from (3.23):

(a) for one perturbation to be satisfactory, k must not exceed 2 per cent,

(b) for $k = 5$ per cent, at least two perturbations are required.

It also appears doubtful that the alternating series (3.23) would converge for values of k greater than 15 per cent.

It must be emphasized that general conclusions regarding convergence and accuracy cannot be based on qualitative conclusions for a particular stress state and a particular type of nonlinearity.

In another paper [23] the authors have examined axisymmetric plane strain boundary value problems for physically nonlinear solids. Proceeding from the single radial displacement equation of equilibrium, the limitations of the perturbation method were found to be consistent with those described in the present paper.

Thus, it appears that, for the present, the use of the perturbation method in the solution of boundary value problems in nonlinear elasticity must be regarded with caution.

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Résumé—Des équations constitutives sont développées pour les solides élastiques supportant des déformations pour lesquelles des inclinaisons de déplacement sont petites mais où la non–linéarité physique complète est permise. L'équation constitutive inclue, comme cas spéciaux, des formes considérées par des auteurs récents; en même temps, des effets plus généraux sont incorporés, en particulier l'accouplement entre des composants de contrainte volumétrique et déviatoire.

Quelques cas d'état de simple déformation sont examinés et le problème de l'élastostatique plane formulés, avec la solution d'un problème exemple par perturbation technique.

Zusammenfassung—Grundlegende Gleichungen für elastische Körper mit aufrechterhaltender Verformung wurden entwickelt mit kleinen Verschiebungsgefällen, in welchen aber vollständige physikalische Nichtlinearität erlaubt ist. Die grundlegende Gleichung enthält Formen, besonders berücksichtigt bei modernen Autoren; darüberhinaus sind allgemeinere Wirkungen mit einbeschlossen, insbesondere die Vereinigung zwischen volumetrischen und abweichenden Beanspruchungskompenenten.

Einige einfache Verformungszustände werden untersucht und das flächenelastostatische Problem wird, zusammen mit der Lösung eines Problembeispieles bei Störungstechniken formuliert.

Абстракт—Выработаны конститутивные уравнения для уиругих твёрдых тел, противостоящих деформации, для которых градиенты смещения малы, но где допускается полная физическая нелинейность. Конститутивное уравнение включает специальные случаи рассмотренные медавно

разными авторами: в то же самое время включено ещё больше общих результатов работы, в особенности по объединению между объёмными и отклоняющимися составными частями деформации. Исследуются некоторые простые состояния деформации и сформулирована зластостатическая проблема плоскости совместно с решением примерной проблемы техникой пертурбации.